□-decay theory
\( \alpha \)-particle penetration through Coulomb Barrier

- **Goal**: *estimate the parent lifetime for \( \alpha \)-decay*
- **Assume** an \( \alpha \)-particle is formed in the parent nucleus \( \alpha \)
- Parent nucleus = \( \alpha \)-particle + daughter nucleus
- \( \alpha \)-decay \( \alpha \) \( \alpha \)-particle must “tunnel” through the Coulomb barrier from \( R \) (nuclear matter radius) to \( b \) (called the “turning point”).
Alpha Decay for Rn-222
\[ -\text{-particle penetration through Coulomb Barrier} \]

- **Goal:** estimate the parent lifetime for \[ -\text{-decay} \]
- Consider the \[ -\text{-particle} \] moving in the potential of the daughter nucleus.
- For \( r > R \), this is a central force problem with

\[
Q = T_{\Box} + T_D \quad ; \quad T_{\Box} = \frac{1}{2} m_{\Box} v_{\Box}^2 \quad ; \quad T_D = \frac{1}{2} m_D v_D^2
\]

\[
Q = T_m = \frac{1}{2} m v_{rel}^2 \quad ; \quad m = \frac{m_{\Box} m_D}{m_{\Box} + m_D}
\]

\[ E = Q \quad \text{Motion of a reduced mass} \quad \Box -\text{particle relative to center of daughter nucleus} \]
□-particle penetration through Coulomb Barrier

- **Goal:** estimate the parent lifetime for □-decay
- Assume that the rate of □-decay per nucleus (□) can be obtained as
  - □ = (probability for □-particle to penetrate the barrier) • (number of hits on the boundary per sec.)
  - (probability for □-particle to penetrate the barrier) = “transmission coefficient” T
\[ \text{-particle penetration through Coulomb Barrier} \]

- (probability for \[ \text{-particle to penetrate the barrier} \) = “transmission coefficient” \( T \)
- To calculate \( T \) for the Coulomb barrier -- consider the \[ \text{-particle} \) transmission through a one-dimensional rectangular barrier.
- From chapter 2, we have the transmission coefficient \( T \) for (simple) case --
1-dimensional rectangular barrier

\[ V_o \]

\[ E \]

\[ 2a \]

\[ 0.0 \]
\[
T = \frac{(2k\Box)^2}{\left(k^2 + \Box^2\right)^2 \sinh^2 2\Box a + (2k\Box)^2}
\]

\[
k = \frac{\sqrt{2mE}}{\hbar}; \quad \Box = \frac{\sqrt{2m(V_o \Box E)}}{\hbar}
\]

\[V_o >> E; \quad \Box a \text{ is “large”}\]

\[
\sinh^2 2\Box a = \left[ e^{2\Box a} \Box e^{2\Box a} \right]^2
\]

\[
\sinh^2 2\Box a \Box e^{4\Box a}
\]

\[
\left(k^2 + \Box^2\right)^2 e^{4\Box a} + (2k\Box)^2 \Box \left(k^2 + \Box^2\right)^2 e^{4\Box a}
\]

\[
T \Box \frac{(2k\Box)^2}{\left(k^2 + \Box^2\right)^2 e^{4\Box a}} = \frac{(2k\Box)^2 e^{4\Box a}}{(k^2 + \Box^2)^2} \quad e^{4\Box a} \text{ important!!}
\]
Coulomb barrier penetration

- Consider penetration through Coulomb barrier as a sequence of rectangular barriers.
- Total transmission coefficient is

\[ T = T_1 \cdot T_2 \cdot T_3 \cdot T_4 \cdots = \prod_i T_i \]

\[ \ln T = \sum_i \ln T_i \]
Coulomb barrier penetration

\[ LnT_i = 2i(2a) + 2Ln \left( \frac{2(ka)(i\lambda a)}{ka} \right) + Ln \text{ term small} \]

\[ LnT_i = 2i(2a) \]

\[ LnT_i = 2i(\lambda x) ; \lim_{x \to 0} \]

\[ LnT_{barrier} = 2 \frac{\sqrt{2m[V(x) - E]}}{\hbar} dx \]
Coulomb barrier penetration

\[ \ln T = \int_{\text{barrier}} 2 \frac{\sqrt{2m[V(x) - E]}}{\hbar} \, dx \]

- We have assumed \( V(x) \gg E \)
- Where in \( x \) is this approximation less good?

\[ T = e^{-G} \]

\[ G = 2 \frac{2m}{\hbar^2} \left[ \frac{b}{4} \right]^{1/2} Z_1 Z_2 e^2 \frac{E}{4b r} \left[ \frac{1}{2} \right]^{1/2} \, dr \]

\( Z_1 \) daughter \( Z_2 = 2 \)
Coulomb barrier penetration

\[ E = \frac{Z_1 Z_2 e^2}{b} \]

\[ \frac{1}{r - b} \cdot \frac{1}{1/b} = \sqrt{b} \cos^1 \frac{R}{b} \]

If \( E << V(R) \) Low Energy \(-\)-particle \( b >> R \)

\[ \frac{R}{b} \quad 0 \quad \cos^1 \frac{R}{b} \quad \frac{R}{b} \]

\[ \frac{R^2}{b^2} \ll \frac{R}{b} \]
Coulomb barrier penetration

\[ G = \frac{2mc^2Z_1Z_2e^2b^{1/2}}{4\sqrt{b}\hbar^2c^2} \]

\[ b = \frac{Z_1Z_2e^2}{4\sqrt{b}E} \]

\[ E = \frac{1}{2}mv_{in} \]

\[ m = \frac{m\Box m_D}{m\Box + m_D} \]

\[ E = Q \]

\[ \frac{R}{b} = \frac{Q}{B} \]

You can easily show this ratio is true.
Coulomb barrier penetration

\[ G = \frac{2mc^2}{\hbar^2 c^2 Q} \left( \frac{Z_1 Z_2 e^2}{4 \frac{Q}{B}} \right)^{1/2} \]

\[ T = e^{-G} \]

\[ T = \exp \left( -2 \frac{2mc^2}{\hbar^2 c^2 Q} \left( \frac{Z_1 Z_2 e^2}{4 \frac{Q}{B}} \right)^{1/2} \right) \]
Barrier hit frequency

• For $r < R$, $\square$-particle moves with $v_{in}$

$$f = \frac{v_{in}}{R}$$

*Frequency of hits on Coulomb barrier*

$$v_{in} = \sqrt{\frac{2T_{in}}{m}}$$

*Velocity of $\square$-particle inside daughter*

$$T_{in} = Q \square V_o$$

*Remember: $V_o$ is negative!*

$$\square = f \ T$$

*$\square$-particle decay constant - probability per unit time for decay*
-particle decay lifetime

\[ t_{1/2} = \frac{\ln 2}{\frac{2mc^2}{\hbar^2 c^2 Q} \exp \left( \frac{2mc^2}{\hbar^2 c^2 Q} \frac{Z_1 Z_2 e^2}{4Qb} \right) + \frac{Rc}{2\sqrt{2(Q\Box V_0)/mc^2}}} \]
\[ t_{1/2} = \frac{\ln 2}{2} \cdot \frac{R_c}{\sqrt{2/Q \cdot V_o}} \cdot \exp \left[ +2 \cdot \frac{2mc^2}{\hbar c^2 Q} \cdot \frac{1/2}{2} \cdot \frac{Z_1 Z_2 e^2}{4 \cdot \overline{\overline{\cdot}}} \cdot \frac{1/2}{2} \cdot \frac{Q}{B} \right] \]

\( Q \) is measured for \( \alpha \)-decay

\( m \) is calculated for \( \alpha \) and daughter; \( Z_1 \) & \( Z_2 \) are known

\( R \) is assumed to be \( R_0 A^{1/3} \) \( \alpha \) Calculate \( B \)

\( V_o \) is taken to be \( \approx 35 \text{ MeV} \) (from other studies)

\( A \) is for parent nucleus; may not give best radius!
\(\alpha\)-particle decay lifetime

• **Assumptions:**
  - \(\alpha\)-particle *preformation* in nucleus; did not estimate the rate for this from Fermi GR -- involves
  \[ V_{fi} = \int f V_{a} i dV \]

*Final nuclear state* of \(\alpha\)-particle + daughter nucleus

*Initial nuclear state* of parent nucleus

\(V_a\) causes transition from \(i\) to \(f\)
\[\alpha\]-particle decay lifetime

- **Assumptions:**
  - Assumed nucleus *is a sphere*; many large-A nuclei are deformed (ellipsoids)
  - Assumed the \(\alpha\)-particle takes off zero angular momentum, i.e., \(\ell = 0\)